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COMMENT

Asymptotic number of needles in Laplacian growth

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Abstract. Using linear stability analysis we argue that the asymptotic number of needles growing out of a centre in a Laplacian process is two in all dimensions. Special arrangements in two dimensions can be solved by conformal mapping.

During the past few years the asymptotic shape of Laplacian growth patterns has attracted much attention (see Kertész 1987). One of the important questions is the asymptotic number of main arms, $n_{\max}$, in diffusion-limited aggregation (DLA). Ball (1986) argued, using stability analysis and scaling assumptions, that $(n_{\max}/2-1)\times(D-1)=1$ in two dimensions where D is the fractal dimension of the cluster. This expression leads to $n_{\max}$ being between 4.9 and 6. Meakin and Vicsek (1987) found $n_{\max}\approx 4$-5 in a continuum DLA where radial anisotropy was introduced by a relaxation procedure. Szép and Lugosi (1986) described the growth of two-dimensional needles in a Laplacian process which can be considered as the model of the extreme anisotropic limit. The purpose of the present comment is to investigate $n_{\max}$, the asymptotic number of needles, by means of linear stability analysis.

Let us first consider $n=2j$ $(j=1,2,\ldots)$ needles of length l in two dimensions growing out of a centre; the angles between two neighbouring needles are π/j. In order to carry out linear stability analysis we suppose that every second needle is longer by a small increment h.

The growth of this pattern is described by the following equations:

$$\Delta u=0 \tag{1}$$

$$v_n=-\tilde{D}(\nabla u)\hat{n} \tag{2}$$

$$u|_{\Gamma}=0 \tag{3}$$

where u is a Laplacian field, Γ is the boundary of the pattern, $\hat{n}$ its normal vector and v_n is the normal velocity of the interface; $\tilde{D}$ corresponds to the diffusion constant. Equations (1)-(3) appear in the context of dendritic growth, viscous fingering, dielectric breakdown and diffusion-limited aggregation. Equation (3) represents the case of vanishing surface tension.

To fix the boundary condition far from the pattern we can write

$$\lim_{|r|\to\infty} u(r)/\log|r|=C \tag{4}$$

where $C<0$ corresponds to the pattern forming case (e.g. undercooling in dendritic solidification).

Let us now introduce a complex analytic function $f(z)$ which maps the field around the perturbated star out of the unit circle. Neglecting the width of the needles we obtain

$$f(z)=\left\{\frac{2}{A+B}[z^{n/2}-\tfrac{1}{2}(A-B)]+\left[\left(\frac{2}{A+B}[z^{n/2}-\tfrac{1}{2}(A-B)]\right)^{2}-1\right]^{1/2}\right\}^{2/n} \tag{5}$$

where $A=(l+h)^{n/2}$ and $B=l^{n/2}$. Then the solution of (1) with the boundary condition (3), (4) is

$$u(r,\varphi)=C\log|f(z)| \tag{6}$$

where $z=r\,e^{i\varphi}$ represents a point in the complex plane.

In order to avoid divergences at the tips we rewrite the boundary equation (2) in terms of finite differences:

$$\frac{dh}{dt}\approx-\frac{\tilde{D}}{a}[u(r=l+h+a,\varphi=0)-u(r=l+a,\varphi=\pi/j)] \tag{7}$$

where the cut-off a has been introduced (Szép and Lugosi 1986) and we have supposed that there is a needle in the $\varphi=0$ direction.

After linearising the right-hand side of (7) and introducing the notation $h(t)\sim e^{\omega_n t}$ we obtain

$$\omega_n=-\frac{\tilde{D}C}{2l}\frac{n-2}{n}\frac{1}{a}\frac{\gamma}{1+\gamma} \tag{8}$$

with $\gamma=na/l+(na/l)^{1/2}$. We see that $n>2$ is unstable, $n_{max}=2$ marginally stable. Here we mention that a similar analysis of the $n=4$ case with a perturbation on one needle only leads also to instability.

Although the above derivation is based on a special two-dimensional arrangement, we think that the result is more general: it is valid in arbitrary dimensions. If n_{max} were greater than two, there should be a range in n which would be stable. On the other hand, by changing the sign of C, these n values would become unstable (cf equation (8)) but in this case the driving force for such an instability is lacking.

Finally we mention the possible connection of our result with anisotropic DLA. Anisotropy seems to influence the asymptotic shape of DLA clusters in a relevant way (Meakin 1986, Kertész and Vicsek 1986). However, it is not clear whether it breaks through entirely resulting finally in arms with overall shape of needles (Kertész and Vicsek 1986, Kertész *et al* 1986) or a balance of anisotropy and fluctuations develops leading to fractal structures (Thompson (1987); for a somewhat different model see Nittmann and Stanley (1986)). In the first case our analysis could be immediately relevant for DLA also. In the second it is possible that n varies with the strength of anisotropy and our result would correspond only to the large anisotropy limit; then the mechanism of mode selection remains to be understood.

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